MATHEMATICS

B.Sc, Part-I, Paper II

Topic → CURVATURE

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Cartesian formula for the radius of curvature — Explicit functions

Let the equation of the curve be $y = f(x)$.

We know that $\tan\psi = \frac{dy}{dx}$.

Differentiating this with r.t.s, we get

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds}.$$

$$\text{or, } \sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cos\psi \qquad \left[\because \frac{dx}{ds} = \cos\psi\right]$$

$$\text{or, } \sec^2\psi \frac{1}{\rho} = \frac{d^2y}{dx^2} \cos\psi \qquad \left[\because \frac{ds}{d\psi} = \rho\right]$$

$$\text{or, } \frac{\sec^2\psi}{\frac{d^2y}{dx^2}} = \rho$$

$$\text{or, } \rho = \frac{(\sec^2\phi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \qquad \left[\because \frac{dy}{dx} = \tan\psi\right]$$

$$= \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx} \cdot y_2 = \frac{d^2y}{dx^2}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_2 \neq 0 \qquad \text{--- (A)}$$

If y is the independent variable, it can be similarly shown that

$$= \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = \frac{\{1+x_1^2\}^{3/2}}{x_2} \qquad \text{--- (B)}$$

where $x_1 = \frac{dx}{dy}$ $x_2 = \frac{d^2x}{dy^2}$, and $x_2 \neq 0$.

Remark :- (i) Formula (A) fails when $\frac{dy}{dx}$ is infinite that is when the tangent at P is parallel to the y-axis. Then the formula would be found useful for the curve as $x = F(y)$.

(ii) The radius of curvature ρ, is positive or negative according as y_2 is positive or negative that is according as the curve is concave or convex upwards

Parametric Formula for the radius of curvature

Let the equation of the curve be

$$x = \phi(t),\ y = \psi(t)$$

We have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \quad \left[\because y' = \frac{dy}{dt},\ x' = \frac{dx}{dt}\right]$$

$$\therefore \frac{dy}{dx} = \frac{y'}{x'} \ (x' \neq 0)$$

Differentiating (1) both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{x'^2} \cdot \frac{1}{x'} \quad \left[\because x'' = \frac{d^2x}{dt^2},\ y'' = \frac{d^2y}{dt^2}\right]$$

$$\frac{d^2y}{d^2x} = \frac{x'y'' - y'x''}{x'^3} \quad \text{———— (2)}$$

Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (1) and (2) respectively i.e.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \text{ we have then}$$

$$\rho = \frac{\left[1 + \frac{y'^2}{x'^2}\right]^{3/2}}{\frac{x'y'' - y'x''}{x'^3}} = \frac{\frac{(x'^2 + y'^2)^{3/2}}{x'^3}}{\frac{x'y'' - y'x''}{x'^3}}$$

$$\therefore \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}, \text{ where } x'y'' - y'x'' \neq 0$$

Radius of curvature for cartesian curves.
Implicit functions:

Let the equation of the curve be
$$x = \phi(s)\ y = \psi(s)$$
We know that
$$\cos\psi = \frac{dx}{ds}\ ;\ \sin\psi = \frac{dy}{ds}$$
Differentiating $\cos\psi = \frac{dx}{ds}$ with respect to s, have
$$-\sin\psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2} \quad \text{------ ----(1)}$$
Again differentiating $\sin\psi = \frac{dv}{ds}$ with respect to s we have
$$\cos\psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2} \quad \text{—— ——(2)}$$
From (1) we have
$$-\frac{dy}{ds} \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2}$$
From (2) we have
$$\frac{dx}{ds} \cdot \frac{1}{\rho} = \frac{d^2y}{ds^2}$$
$$\therefore\ \rho = \frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}} = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}}$$
Squaring and then adding, we have
$$\frac{1}{\rho^2} = \left(\frac{d^2x}{dx^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \quad \left[\because \left(\frac{dv}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2 = 1\right]$$

Radius of curvature for cartesian curves.
Implicit Functions

Let f(x, y) be the implicit equation of the curve.
$$\frac{dy}{dx} = -\frac{f_x}{f_y}\ (f_y \neq 0)$$
i.e. $f_x + f_y \dfrac{dy}{dx} = 0$ —— — — — —.(1)

Differentiating w.r.t. x, we get

$$f_{xx} + f_{xy}\frac{dy}{dx} + f_{yx}\cdot\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\cdot\frac{d^2y}{dx^2} = 0 \quad \text{----(2)}$$

Substituting the value of $\frac{dy}{dx}$ from (1) and assuming that $f_{yx} = f_{xy}$ in equation (2), we get

$$f_{xx} + 2f_{xy}\left(-\frac{f_x}{f_y}\right) + f_{yy}\left(-\frac{f_x}{f_y}\right)^2 + f_y\frac{d^2y}{dx^2} = 0$$

or, $$f_{xx} - 2f_{xy}\frac{f_x}{f_y} + f_{yy}\left(\frac{f_x}{f_y}\right)^2 + f_y\frac{d^2y}{dx^2} = 0$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{f_y^2 . f_{xx} - 2f_{xy}.f_xf_y + f_{yy}f_x^2}{f_y^3}$$

Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$ we get

$$\rho = \frac{\left(1+\frac{f_x^2}{f_y^2}\right)^{3/2}}{\frac{f_y^2 f_{xx} - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}}$$

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

taking positive sign.

Remark: The form of this formula show that the expression for ρ will retain the same form at points where

$f_y = 0$ but $f_x \neq 0$